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Formulation and integration of the standard linear viscoelastic solid with fractional order rate laws

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Abstract

A physically sound three-dimensional anisotropic formulation of the standard linear viscoelastic solid with integer or fractional order rate laws for a finite set of the pertinent internal variables is presented. It is shown that the internal variables can be expressed in terms of the strain as convolution integrals with kernels of Mittag–Leffler function type. A time integration scheme, based on the Generalized Midpoint rule together with the Grünwald algorithm for numerical fractional differentiation, for integration of the constitutive response is developed. The predictive capability of the viscoelastic model for describing creep, relaxation and damped dynamic responses is investigated both analytically and numerically. The algorithm and the present general linear viscoelastic model are implemented into the general purpose finite element code Abaqus. The algorithm is then used together with an explicit difference scheme for integration of structural responses. In numerical examples, the quasi-static and damped responses of a viscoelastic ballast material that is subjected to loads simulating the overrolling of a train are investigated.

1. Introduction

Modeling of viscoelastic response has a long tradition for describing a variety of phenomena (such as creep, relaxation and energy dissipation or damping) in structural analysis. One particular issue, which has been discussed extensively, is how complex the linear viscoelastic model must be, i.e., what is the minimum number of material parameters that is required for an accurate description of the observed material behavior?

Most engineering materials exhibit a weak frequency dependence of the damping characteristics, which is difficult to describe with classic linear viscoelasticity that is based on integer rate laws for the pertinent internal variables. However, if the integer time derivative is replaced by a fractional order derivative operator, the number of parameters to describe damping in an accurate fashion

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can be significantly reduced. It has been argued, see Bagley and Torvik (1983), that it is sufficient to use as few as four parameters for the uniaxial stress situation (two "elastic" constants, one relaxation constant and the non-dimensional fractional order of differentiation). The reason for this is that the Fourier transforms of integer derivatives exhibit a frequency dependence that is proportional to integer order of differentiation, while fractional derivatives exhibit a frequency dependence that is proportional to the fractional order of differentiation. This is believed to be the main features of using fractional derivative operators in this context. Indeed, the linear viscoelastic model together with fractional derivatives have shown to be very flexible also for describing quasistatic response, such as creep and stress relaxation, see Enelund and Olsson (1995).

Although it is possible, at least in theory, to use transform techniques for evaluating the structural response for linear viscoelasticity, it appears useful in practice to employ numerical integration in time, in particular for evaluating dynamic response. Padovan (1987) presents time-integration algorithms for calculating responses of viscoelastic structures governed by a linear viscoelastic model described by a single equation involving fractional derivative operators acting on both stresses and strains (without employing the concept of internal variables). The formulation of viscoelasticity used in the present study is more general and leads to well-posed initial value problems when incorporated into a framework for structural analysis.

The paper is organized as follows: a physically sound formulation of the Standard Linear Viscoelastic Solid with integer as well as fractional order rate laws for a finite set of internal variables is presented. A time integration scheme, based on the Generalized Midpoint rule, is employed for integrating the constitutive response. In a few numerical examples, we consider the quasistatic as well as the dynamic response of a layer of viscoelastic ballast material that is subjected to overrolling of high speed trains.

1.1. Notation

Regular italic characters denote scalar quantities, bold-face italic characters denote vectors and second order tensors (such as strain $\boldsymbol{\varepsilon}$ and stress $\boldsymbol{\sigma}$), while bold-face calligraphic characters are used to denote forth-order tensors (such as the identity tensor $\mathcal{I}$ and the elastic stiffness modulus tensor $\mathcal{E}^{e}$).

"Open product" is denoted $\otimes$. "Scalar product" is denoted $\cdot$, while "double scalar product" is denoted:, as in this example

$$|\boldsymbol{\sigma}|^2 = \boldsymbol{\sigma}:\boldsymbol{\sigma} = \sigma_{ab}\sigma_{ab} \tag{1}$$

The subscript dev stands for deviator, e.g., the stress deviator is defined as

$$\boldsymbol{\sigma}_{\text{dev}} = \boldsymbol{\sigma} - \tfrac{1}{3}\sigma_{\text{vol}}\boldsymbol{\delta} \quad \text{with}\, \sigma_{\text{vol}} = \boldsymbol{\delta}:\boldsymbol{\sigma} \tag{2}$$

where $\boldsymbol{\delta}$ is the second order identity (or Kronecker's delta) tensor.

A superposed dot $(\dot{\ })$ denotes integer differentiation with respect to time, e.g.,

$$\dot{\mathbf{u}} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \tag{3}$$

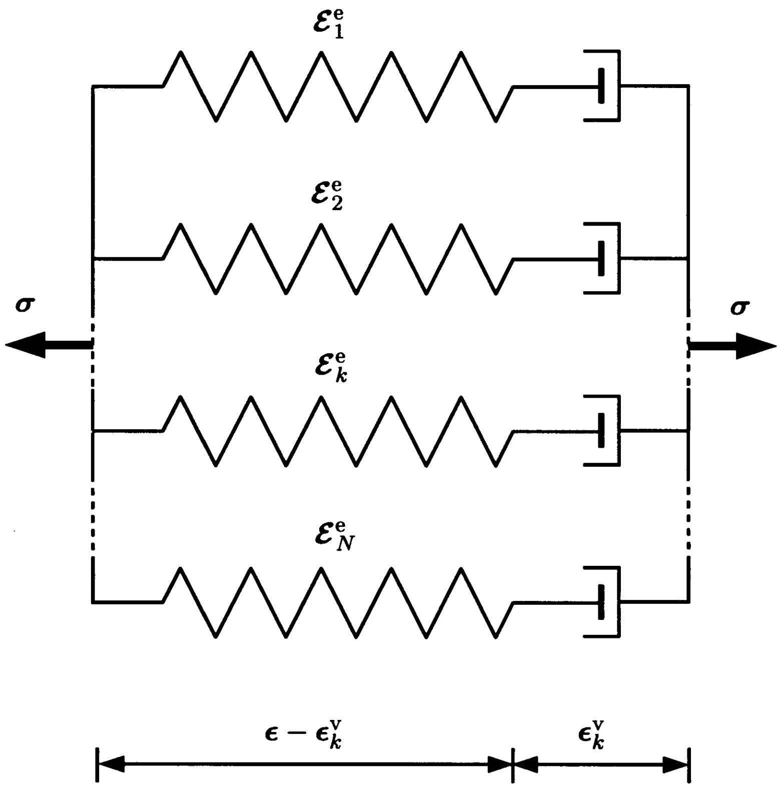

Fig. 1. Mechanical representation of the Linear Standard Viscoelastic model.

2. Linear standard viscoelastic model-anisotropy

2.1. Preliminaries

The classical Linear Standard Viscoelastic model consists of N Maxwell-chains coupled in parallel (see Fig. 1), each one of which is associated with the elastic stiffness tensor $\mathscr{E}_k^e$ and the "elastic strain" tensor $\boldsymbol{\varepsilon}_k^e$ defined as

$$\boldsymbol{\varepsilon}_k^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_k^v(q_{k(1)}, q_{k(2)}, \ldots, q_{k(M)}), \quad k = 1, 2, \ldots, N \tag{4}$$

where $\boldsymbol{\varepsilon}_k^v$ is the viscous (or dissipative) strain tensor. Each $\boldsymbol{\varepsilon}_k^v$ is a function of M scalar internal variables $q_{k(\beta)}$, $\beta = 1, 2, \ldots, M$, with $M = \dim(\boldsymbol{\varepsilon})$, which represent the dissipative mechanisms. It is clear that the explicit choice of these functional relationships and the rate laws for $q_{k(\beta)}$ are crucial for defining the predictive capability of the resulting constitutive equations.

Upon generalizing the expression for the uniaxial stress we express the free energy Ψ (per unit volume) as

$$\Psi = \frac{1}{2}\sum_{k=1}^{N} \boldsymbol{\varepsilon}_k^e : \mathscr{E}_k^e : \boldsymbol{\varepsilon}_k^e = \frac{1}{2}\sum_{k=1}^{N} (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_k^v) : \mathscr{E}_k^e : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_k^v) \tag{5}$$

From the Clausius–Duhem-Inequality (CDI),[1] we then obtain the constitutive relation for the stress $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = \sum_{k=1}^{N} \mathscr{E}_k^{\mathrm{e}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_k^{\mathrm{v}}) = \sum_{k=1}^{N} \boldsymbol{\sigma}_k^{\mathrm{v}} \tag{6}$$

where $\boldsymbol{\sigma}_k^{\mathrm{v}}$ are the viscous stresses (that are energy conjugates to $\boldsymbol{\varepsilon}_k^{\mathrm{v}}$), which are given as

$$\boldsymbol{\sigma}_k^{\mathrm{v}} = -\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}_k^{\mathrm{v}}} = \mathscr{E}_k^{\mathrm{e}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_k^{\mathrm{v}}), \quad k = 1, 2, \ldots, N \tag{7}$$

Remark: The initial elastic stiffness (when $\boldsymbol{\varepsilon}_k^{\mathrm{v}} = 0$) is given as $\mathscr{E}^{\mathrm{e}} = \Sigma_{k=1}^{N} \mathscr{E}_k^{\mathrm{e}}$, which is the elastic response at very high loading rate.

Henceforth, we restrict the model by assuming that all $\mathscr{E}_k^{\mathrm{e}}$ are coaxial tensors. This means that they possess the same orthonormal set of second order eigentensors $\boldsymbol{\varphi}_\beta$, $\beta = 1, 2, \ldots, M$, corresponding to the eigenvalues $E_{k(\beta)} > 0$, in the following sense:

$$\mathscr{E}_k^{\mathrm{e}} : \boldsymbol{\varphi}_\beta = E_{k(\beta)} \boldsymbol{\varphi}_\beta \quad \text{with } |\boldsymbol{\varphi}_\beta| = 1, \quad \beta = 1, 2, \ldots, M \tag{8}$$

Hence, the spectral decomposition of $\mathscr{E}_k^{\mathrm{e}}$ is

$$\mathscr{E}_k^{\mathrm{e}} = \sum_{\beta=1}^{M} E_{k(\beta)} \boldsymbol{\varphi}_\beta \otimes \boldsymbol{\varphi}_\beta \tag{9}$$

We shall now choose $\boldsymbol{\varepsilon}_k^{\mathrm{v}}$ as

$$\boldsymbol{\varepsilon}_k^{\mathrm{v}} = \sum_{\beta=1}^{M} q_{k(\beta)} \boldsymbol{\varphi}_\beta, \quad k = 1, 2, \ldots, N \tag{10}$$

and it follows from the CDI that the dissipative stress quantities, conjugated to $q_{k(\beta)}$, are given as

$$Q_{k(\beta)} = -\frac{\partial \Psi}{\partial q_{k(\beta)}} = E_{k(\beta)} (\boldsymbol{\varphi}_\beta : \boldsymbol{\varepsilon} - q_{k(\beta)}) \tag{11}$$

where the spectral properties in eqn (8) were used.

Remark: Upon comparing eqn (11) with eqn (7), it appears that

$$\boldsymbol{\sigma}_k^{\mathrm{v}} = \sum_{\beta=1}^{M} Q_{k(\beta)} \boldsymbol{\varphi}_\beta \tag{12}$$

in complete analogy with the definition of $\boldsymbol{\varepsilon}_k^{\mathrm{v}}$ in eqn (10).

[1] With CDI we refer to the classical entropy inequality in its spatially (and temporally) strong form. A temporally weak form was suggested by Day (1968) and Day (1969), which is henceforth referred to as the Day Inequality (DI).

2.2. Integer order rate laws

In complete analogy with the uniaxial version of the Linear Standard model, we choose the rate equations for $q_{k(\beta)}$ as the uncoupled equations (this is consistent with classic linear viscoelasticity)

$$\dot{q}_{k(\beta)} = \frac{1}{E_{k(\beta)}\tau_{k(\beta)}} Q_{k(\beta)} = \frac{1}{\tau_{k(\beta)}}(\boldsymbol{\varphi}_\beta : \boldsymbol{\varepsilon} - q_{k(\beta)})$$
$$q_{k(\beta)}(0) = 0, \quad k = 1, 2, \ldots, N, \quad \beta = 1, 2, \ldots, M \tag{13}$$

where $\tau_{k(\beta)} > 0$ are the relaxation times associated with each dissipative mechanism. For the constitutive model to reproduce the solid behavior with finite long time elastic modulus tensor $\mathscr{E}^{\mathrm{e}}_{(\infty)}$, at least one set of relaxation times $\tau_{k(\beta)}$, $\beta = 1, 2, \ldots, M$, for a particular Maxwell chain k, must be infinite. If $\tau_{N(\beta)} = \infty$ for $\beta = 1, 2, \ldots, M$, then $\mathscr{E}^{\mathrm{e}}_{(\infty)} = \mathscr{E}^{\mathrm{e}}_N$. However, in order to retain formal simplicity, we shall not make any *a priori* assumptions as to the magnitude of $\tau_{k(\beta)}$.

It follows immediately that the model is thermodynamically admissible in the classical sense, sine the rate of energy dissipation D is non-negative, i.e.,

$$D \equiv \sum_{k=1}^{N} \sum_{\beta=1}^{M} Q_{k(\beta)} \dot{q}_{k(\beta)} = \sum_{k=1}^{N} \sum_{\beta=1}^{M} \frac{1}{E_{k(\beta)}\tau_{k(\beta)}} (Q_{k(\beta)})^2 \geqslant 0 \tag{14}$$

As to the weaker inequality, DI, it is expressed in the present context as a statement of non-negative energy dissipation up to the current time $t > 0$, i.e.,

$$W_{\mathrm{D}} = \int_0^t D(s)\,\mathrm{d}s \geqslant 0 \tag{15}$$

It is then assumed that $q_{k(\beta)}$ have homogeneous initial condition. Hence, the stronger CDI expressing the condition of nonnegative dissipative power is a sufficient condition for DI, while the inverse does not hold.

2.3. Fractional order rate laws

The only difference as compared to the classical model is that the rate law, eqn (13), is replaced by an expression involving fractional derivative operators of order $\alpha \in (0, 1)$. A suitable definition of fractional differentiation of a function $y(t)$ is defined as follows (see Gel'fand and Shilov, 1964; Oldham and Spanier, 1974): first we define $\mathrm{D}^{-(1-\alpha)}$ (which is a fractional integration) as the convolution

$$\mathrm{D}^{-(1-\alpha)} y(t) \equiv \int_0^t \Phi_{(1-\alpha)}(t - \hat{t}) y(\hat{t})\,\mathrm{d}\hat{t} \tag{16}$$

where

$$\Phi_{(1-\alpha)}(t) = \frac{t_+^{-\alpha}}{\Gamma(1-\alpha)} \quad \text{with } t_+ = \begin{cases} t & t > 0 \\ 0 & t < 0 \end{cases} \tag{17}$$

and Γ is the gamma function. The integral in eqn (16) is normally convergent. Hence, from eqn (16) a convergent expression for the α-order fractional derivative operator D^α is given as (for details we refer to Oldham and Spanier, 1974)

$$\begin{aligned} \mathrm{D}^\alpha y(t) &\equiv \mathrm{D}^1\,\mathrm{D}^{-(1-\alpha)} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \Phi_{(1-\alpha)}(t-\hat{t})y(\hat{t})\,\mathrm{d}\hat{t} \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{y(\hat{t})}{(t-\hat{t})^\alpha}\,\mathrm{d}\hat{t} \end{aligned} \tag{18}$$

and it is noted, in particular, that, D^α is an ordinary derivative operator that requires initial conditions on $y(t)$. It is possible to use the definition in eqn (16) for fractional differentiation, but the integral is then normally divergent and has to be suitably regularized. We are now in the position to propose the fractional order rate law

$$\begin{aligned} \mathrm{D}^{\alpha_{k(\beta)}} q_{k(\beta)} &= \frac{1}{E_{k(\beta)}(\tau_{k(\beta)})^{\alpha_{k(\beta)}}} Q_{k(\beta)} = \frac{1}{(\tau_{k(\beta)})^{\alpha_{k(\beta)}}}(\boldsymbol{\varphi}_\beta\colon \boldsymbol{\varepsilon} - q_{k(\beta)}) \\ q_{k(\beta)}(0) &= 0, \quad k = 1,2,\ldots,N, \quad \beta = 1,2,\ldots \end{aligned} \tag{19}$$

where $\tau_{k(\beta)}$ can not be interpreted as the most probable relaxation time out of a continuous distribution of relaxation times. The fractional order of differentiation $\alpha_{k(\beta)}$ then plays the role of a distribution parameter for the corresponding distribution of relaxation times, see Enelund and Lesieutre (1995). The use of fractional (instead of integer) order operators in the rate laws can now be motivated by the idea that a whole spectrum of dissipative mechanisms can be included in a single viscous strain having fractional order rate laws for the corresponding internal variables $q_{(\beta)}$. It can be trivially shown that

$$q_{k(\beta)}(0) = 0 \rightsquigarrow \mathrm{D}^{-(1-\alpha_{k(\beta)})} q_{k(\beta)}(0) = 0, \quad 0 < \alpha_{k(\beta)} < 1 \tag{20}$$

which is the formal initial condition to eqn (19) rather than $q_{k(\beta)}(0) = 0$. Note that, the operator in eqn (20) is a (fractional) integral operator. However, $q_{k(\beta)}(0) = 0$ is, from a physical point of view, the relevant initial condition.

2.3.1. Convolution integral form

By applying a Laplace transform and a subsequent inversion to the rate laws [eqn (19)] with the corresponding initial conditions in eqn (20), we may express the internal variables $q_{k(\beta)}$ in terms of the strain tensor $\boldsymbol{\varepsilon}$ as convolution integrals with singular kernels of Mittag–Leffler function type (see Enelund and Olsson, 1995). Using the convolution integral description of the internal variables, we reformulate the constitutive relations for the dissipative stress quantities $Q_{k(\beta)}$ as

$$Q_{k(\beta)}(t) = E_{k(\beta)}\left(\boldsymbol{\varphi}_\beta\colon \boldsymbol{\varepsilon}(t) - \int_0^t f_{k(\beta)}(t-\hat{t})(\boldsymbol{\varphi}_\beta\colon \boldsymbol{\varepsilon}(\hat{t}))\,\mathrm{d}\hat{t}\right)$$

$$= E_{k(\beta)}(\boldsymbol{\varphi}_\beta\colon \boldsymbol{\varepsilon}(t) - (f_{k(\beta)} * (\boldsymbol{\varphi}_\beta\colon \boldsymbol{\varepsilon}))(t)) \tag{21}$$

where $f_{k(\beta)}$ is the memory kernel

$$f_{k(\beta)}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{E}_{\alpha_{k(\beta)}}[-(t/\tau_{k(\beta)})^{\alpha_{k(\beta)}}]), \quad t > 0 \tag{22}$$

Here E_α is the α-order Mittag–Leffler function, which is defined as (see Bateman, 1955)

$$\mathrm{E}_\alpha(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(1+\alpha k)} \tag{23}$$

We observe that the fractional calculus model represents a fading memory in the rather strict sense that the memory kernels in eqn (22) are strictly decreasing and monotonic functions of time (i.e., $\mathrm{d}f_{k(\beta)}(t)/\mathrm{d}t < 0$). The reason for this is that the Mittag–Leffler function is completely monotonic for $\alpha \in (0, 1]$ and $t > 0$, i.e. (Bateman, 1955)

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}[\mathrm{E}_\alpha(-\mathrm{t})] \geqslant 0, \quad \mathrm{n} = 1, 2, 3, \ldots \tag{24}$$

Consider the special case of $\alpha_{k(\beta)} = 1$; then the memory kernel $f_{k(\beta)}$ becomes

$$f_{k(\beta)}(t) = \frac{1}{\tau_{k(\beta)}} \mathrm{e}^{-t/\tau_{k(\beta)}}, \quad t > 0 \tag{25}$$

which are the exponentially decaying memory functions that correspond to the classical linear viscoelastic model with integer order rate laws.

Consider the convolution term $f^*_{k(\beta)}(\boldsymbol{\varphi}_\beta\colon \boldsymbol{\varepsilon})$ in eqn (21). Enelund and Olsson (1995) showed that its fractional order derivatives satisfies the following simple relation

$$\mathrm{D}^{\alpha_{k(\beta)}}(f_{k_\beta} * (\boldsymbol{\varphi}_\beta\colon \boldsymbol{\varepsilon})) = \frac{1}{(\tau_{k(\beta)})^{\alpha_{k(\beta)}}}(\boldsymbol{\varphi}_\beta\colon \boldsymbol{\varepsilon} - f_{k(\beta)} * (\boldsymbol{\varphi}_\beta\colon \boldsymbol{\varepsilon})), \quad t > 0 \tag{26}$$

This is exactly the fractional order rate law if the convolution integral term is interpreted as an internal variable.

The constitutive relation in eqn (21) can also be written as

$$Q_{k(\beta)}(t) = E_{k(\beta)} \int_{0^-}^{t} G_{k(\beta)}(t - \hat{t})(\boldsymbol{\varphi}_\beta\colon \dot{\boldsymbol{\varepsilon}}(\hat{t}))\, \mathrm{d}\hat{t} \tag{27}$$

with

$$G_{k(\beta)}(t) = \mathrm{E}_{\alpha_{k(\beta)}}[-(t/\tau_{k(\beta)})^{\alpha_{k(\beta)}}], \quad t \geqslant 0 \tag{28}$$

Here $E_{k(\beta)} G_{k(\beta)}(t)$ is the dissipative stress response for a unit step strain tensor applied at time $t = 0$. Taking the lower limit as 0^- in the integral in eqn (27) allows for step discontinuities in the strains at time $= 0$ by the standard relation between the Heaviside step function and the Dirac delta function. The short and long time behavior of the kernels $G_{k(\beta)}$ are (see Bateman, 1955)

$$\lim_{t\to 0^+} G_{k(\beta)}(t) = \lim_{\tau_{k(\beta)}\to\infty} G_{k(\beta)}(t) = 1 \quad \text{and} \quad \lim_{t\to\infty} G_{k(\beta)} = 0$$

Consider the simple case of uniaxial stress ($\beta = 1$ and $\boldsymbol{\varphi} = 1$), for two parallel Maxwell chains with $\tau_2 \to \infty$. The stress relaxation function or the relaxation modulus (i.e., the stress response on a unit strain imposed at time $t = 0$) can then be written as

$$\sigma_{\text{rel}}(t) = E_1\,\mathrm{E}_\alpha[-(t/\tau_1)^\alpha] + E_2, \quad t \geqslant 0 \tag{29}$$

The stress relaxation function is obtained by applying a Laplace transform and a subsequent inverse Laplace transformation to the constitutive response to a unit step strain (for a full derivation see Enelund, 1997). The corresponding instantaneous and long-time responses are

$$\lim_{t\to 0^+} \sigma_{\text{rel}}(t) = E_1 + E_2 = E_{(0)} \quad \text{and} \quad \lim_{t\to 0^+} \sigma_{\text{rel}}(t) = E_2 = E_{(\infty)}$$

as they should be.

The convolution integral forms of viscoelasticity in, e.g., eqns (21) and (27) are often referred to as the Boltzmann (1876) or hereditary models. Singular kernels of Mittag–Leffler functions were first introduced into viscoelasticity by Rabotnov (1980). Koeller (1984) established the relationship between kernels of Mittag–Leffler function type and the fractional calculus model of viscoelasticity.

2.3.2. Thermodynamic admissibility

It is natural to ask if the dissipation inequality as formulated in eqn (14) is satisfied for $\alpha \in (0, 1)$. By a counter example it is simple to show that the CDI is not generally satisfied. To this end, consider again the simple situation of uniaxial stress with the extreme choice $\alpha = 0$, which gives $q_1 = \varepsilon/2$ and $q_2 = 0$. We then obtain

$$D = Q_1\dot{q}_1 + Q_2\dot{q}_2 = \frac{E_1}{4}\varepsilon\dot{\varepsilon} \tag{30}$$

which may take any sign. However,

$$W_{\text{D}} = \int_0^t D(s)\,\mathrm{d}s = \frac{E_1}{8}\varepsilon^2 \geqslant 0 \quad \text{with}\ \varepsilon(0) = 0 \tag{31}$$

We are thus lead to investigate whether DI is satisfied or not. In the general situation, we write the DI inequality as

$$W_{\text{D}} = \sum_{k=1}^{N}\sum_{\beta=1}^{M}\int_0^t Q_{k(\beta)}(s)\dot{q}_{k(\beta)}(s)\,\mathrm{d}s \geqslant 0 \tag{32}$$

By introducing the fractional order rate law for $q_{k(\beta)}$ in eqn (19) together with the definitions in eqn (16) and (18) for fractional order integration and differentiation, respectively, we write the dissipation work for the actual model (corresponding to DI) as

$$W_{\text{D}} = \sum_{k=1}^{N}\sum_{\beta=1}^{M} E_{k(\beta)}(\tau_{k(\beta)})^{\alpha_{k(\beta)}}\int_0^t \mathrm{D}^{\alpha_{k(\beta)}}(\mathrm{D}^1\,\mathrm{D}^{-1}q_{k(\beta)}(s))\,\mathrm{D}^1 q_{k(\beta)}(s)\,\mathrm{d}(s)$$

$$= \sum_{k=1}^{N} \sum_{\beta=1}^{M} E_{k(\beta)}(\tau_{k(\beta)})^{\alpha_{k(\beta)}} \int_0^t \dot{q}_{k(\beta)}(s)(\mathrm{D}^{-(1-\alpha_{k(\beta)})}\dot{q}_{k(\beta)}(s))\,\mathrm{d}s$$

$$= \sum_{k=1}^{N} \sum_{\beta=1}^{M} E_{k(\beta)}(\tau_{k(\beta)})^{\alpha_{k(\beta)}} \int_0^t \dot{q}_{k(\beta)}(s)\left(\int_0^t \Phi_{(1-\alpha_{k(\beta)})}(s-\hat{t})\dot{q}_{k(\beta)}(\hat{t})\,\mathrm{d}\hat{t}\right) ds$$

$$= 2\sum_{k=1}^{N} \sum_{\beta=1}^{M} E_{k(\beta)}(\tau_{k(\beta)})^{\alpha_{k(\beta)}} \int_0^t \int_0^t \Phi_{(1-\alpha_{k(\beta)})}(s-\hat{t})\dot{q}_{k(\beta)}(\hat{t})\dot{q}_{k(\beta)}(s)\,\mathrm{d}\hat{t}\,\mathrm{d}s \geqslant 0 \tag{33}$$

where we used the homogeneous initial conditions on $q_{k(\beta)}$ when composing orders of generalized differintegration. The kernel function $\Phi_{(1-\alpha_{k(\beta)})}(t)$ is only defined for positive values of the argument. In eqn (33) we extend the domain of definition to negative t by assuming $\Phi_{(1-\alpha_{k(\beta)})}(t)$ to be an even function, i.e., $\Phi_{(1-\alpha_{k(\beta)})}(t)$ is replaced by $\Phi_{(1-\alpha_{k(\beta)})}(|t|)$. By use of a table of Fourier transforms of generalized functions (see Gel'fand and Shilov, 1964 p. 359), we obtain the Fourier transform of the kernel function in eqn (33) as

$$\mathscr{F}(\Phi_{(1-\alpha_{k(\beta)})}(|t|)(\omega) = \int_{-\infty}^{\infty} \frac{|t|^{-\alpha_{k(\beta)}}}{\Gamma(1-\alpha_{k(\beta)})}\mathrm{e}^{i\omega t}\,\mathrm{d}t = 2\sin(\alpha_{k(\beta)}\pi/2)|\omega|^{-(1-\alpha_{k(\beta)})} \tag{34}$$

which is an even positive function (distribution) of ω. It then follows from the Bochner-Schwartz theorem (see Reed and Simon, 1975) that $\Phi_{(1-\alpha_{k(\beta)})}(t)$ are functions (distributions) of positive type. Therefore (see Reed and Simon, 1975) the dissipation work W_{D} in eqn (33) is non-negative, since $E_{k(\beta)} > 0$ and $\tau_{k(\beta)} > 0$. We can now conclude that the temporally weak form of the dissipation inequality (DI) is satisfied for arbitrary loadings, while the temporally strong form of the dissipation inequality (CDI) is not satisfied in general.

3. Linear standard viscoelastic model—isotropy

In the case of isotropic viscoelastic response, only two internal variables are required (that correspond to the deviatoric and volumetric responses). Although this may be taken as a trivial starting point, it can also be obtained from the general theory as follows: according to isotropic linear elasticity, the eigenvalues $E_{k(\beta)}$, for each k, are given as

$$E_{k(1)} = 3K_k, \quad E_{k(2)} = E_{k(3)} = \cdots = E_{k(M)} = 2G_k \tag{35}$$

corresponding to the eigentensors

$$\boldsymbol{\varphi}_1 = \frac{1}{\sqrt{3}}\boldsymbol{\delta}, \quad \boldsymbol{\varphi}_\beta \in \mathscr{V}_{\mathrm{dev}} = \{\psi|\psi\colon \boldsymbol{\delta} = \mathbf{0}\}, \quad \beta = 2, 3, \ldots, M \tag{36}$$

Here we have introduced the space of deviatoric second order tensors $\mathscr{V}_{\mathrm{dev}}$, which is spanned by the $M-1$ mutually orthonormal $\boldsymbol{\varphi}_\beta$. Clearly, K_k and G_k are the bulk modulus and shear modulus, respectively, associated with the kth Maxwell chain.

In order to ensure complete viscoelastic isotropy, we also choose

$$\tau_{k(1)} = \tau_k^{\mathrm{K}}, \quad \tau_{k(2)} = \tau_{k(3)} = \cdots = \tau_{k(M)} = \tau_k^{\mathrm{G}} \tag{37}$$

and

$$\alpha_{k(1)} = \alpha_k^{\mathrm{K}}, \quad \alpha_{k(2)} = \alpha_{k(3)} = \cdots = \alpha_{k(M)} = \alpha_k^{\mathrm{G}} \tag{38}$$

Upon splitting $\boldsymbol{\varepsilon}_k^{\mathrm{v}}$ in eqn (10) in its deviatoric and volumetric parts, while using eqn (36), we obtain

$$\boldsymbol{\varepsilon}_k^{\mathrm{v}} = \boldsymbol{\varepsilon}_{k,\mathrm{dev}}^{\mathrm{v}} + \tfrac{1}{3}\varepsilon_{k,\mathrm{vol}}^{\mathrm{v}}\boldsymbol{\delta} \tag{39}$$

with

$$\varepsilon_{k,\mathrm{dev}}^{\mathrm{v}} = \sum_{\beta=2}^{M} q_{k(\beta)}\boldsymbol{\varphi}_\beta \quad \text{and} \quad \varepsilon_{k,\mathrm{vol}}^{\mathrm{v}} = \boldsymbol{\varepsilon}_k^{\mathrm{v}}\colon\boldsymbol{\delta} = \sqrt{3}q_{k(1)} \tag{40}$$

Moreover, from eqn (7) we obtain in a trivial fashion

$$\boldsymbol{\sigma}_k^{\mathrm{v}} = \boldsymbol{\sigma}_{k,\mathrm{dev}}^{\mathrm{v}} + \tfrac{1}{3}\sigma_{k,\mathrm{vol}}^{\mathrm{v}}\boldsymbol{\delta} \tag{41}$$

with

$$\boldsymbol{\sigma}_{k,\mathrm{dev}}^{\mathrm{v}} = 2G_k(\boldsymbol{\varepsilon}_{\mathrm{dev}} - \boldsymbol{\varepsilon}_{k,\mathrm{dev}}^{\mathrm{v}}) \quad \text{and} \quad \sigma_{k,\mathrm{vol}}^{\mathrm{v}} = 3K_k(\varepsilon_{\mathrm{vol}} - \varepsilon_{k,\mathrm{vol}}^{\mathrm{v}}) \tag{42}$$

Finally, we note that the instantaneous shear modulus and bulk modulus are found as

$$G = \sum_{k=1}^{N} G_k \quad \text{and} \quad K = \sum_{k=1}^{N} K_k \tag{43}$$

whereas the long-time counterparts are found as (in the case that $\tau_N^{\mathrm{G}} \to \infty$ and $\tau_N^{\mathrm{K}} \to \infty$)

$$G_{(\infty)} = G_N \quad \text{and} \quad K_{(\infty)} = K_N \tag{44}$$

Because of the assumptions in eqns (37) and (38), we now obtain from eqn (19)

$$\mathrm{D}^{\alpha_k^{\mathrm{G}}}\boldsymbol{\varepsilon}_{k,\mathrm{dev}}^{\mathrm{v}} = \frac{1}{(\tau_k^{\mathrm{G}})^{\alpha_k^{\mathrm{G}}}}(\boldsymbol{\varepsilon}_{\mathrm{dev}} - \boldsymbol{\varepsilon}_{k,\mathrm{dev}}^{\mathrm{v}}), \quad k = 1, 2, \ldots, N, \quad 0 < \alpha_k^{\mathrm{G}} \leqslant 1 \tag{45a}$$

and

$$\mathrm{D}^{\alpha_k^{\mathrm{K}}}\varepsilon_{k,\mathrm{vol}}^{\mathrm{v}} = \frac{1}{(\tau_k^{\mathrm{K}})^{\alpha_k^{\mathrm{K}}}}(\varepsilon_{\mathrm{vol}} - \varepsilon_{k,\mathrm{vol}}^{v}), \quad k = 1, 2, \ldots, N, \quad 0 < \alpha_k^{\mathrm{K}} \leqslant 1 \tag{45b}$$

In order to obtain eqn (45a) we chose $\beta = 1$ in eqn (19), whereas eqn (45b) was obtained upon multiplying by $\boldsymbol{\varphi}_\beta$ and summing all the terms from $\beta = 2$ to $\beta = M$. Here, it was also used that $\mathscr{I}_{\mathrm{dev}}\colon\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{dev}}$,where $\mathscr{I}_{\mathrm{dev}}$ is the fourth-order deviator projection tensor defined by

$$\mathscr{I}_{\text{dev}} = \sum_{\beta=2}^{M} \boldsymbol{\varphi}_\beta \otimes \boldsymbol{\varphi}_\beta = \mathscr{I} - \frac{1}{3}\boldsymbol{\delta} \otimes \boldsymbol{\delta}, \quad \boldsymbol{\varphi}_1 \otimes \boldsymbol{\varphi}_1 = \frac{1}{3}\boldsymbol{\delta} \otimes \boldsymbol{\delta} \tag{46}$$

3.1. Eight parameter model

Due to the flexibility in fitting measured viscoelastic material data to the Linear Standard Model of viscoelasticity when using fractional order rate laws, it is often sufficient to use a single dissipative strain decomposed in a deviatoric and volumetric part. As a consequence, the model requires eight parameters in the isotropic case and it is obtained by using the following set of parameters:

$$N = 2, \quad \tau_2^{\text{K}} \to \infty \quad \text{and} \quad \tau_2^{\text{G}} \to \infty \rightsquigarrow \boldsymbol{\varepsilon}_2^{\text{v}} = \mathbf{0} \tag{47}$$

Upon using the notation $\boldsymbol{\varepsilon}_1^{\text{v}} = \boldsymbol{\varepsilon}^{\text{v}}$ (etc.) we obtain the constitutive relation for the stress ($\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{dev}} + 1/3\sigma_{\text{vol}}\boldsymbol{\delta}$) from eqns (6) and (42) as

$$\boldsymbol{\sigma}_{\text{dev}} = 2(G_1 + G_2)\boldsymbol{\varepsilon}_{\text{dev}} - 2G_1\boldsymbol{\varepsilon}_{\text{dev}}^{\text{v}} \tag{48a}$$

$$\sigma_{\text{vol}} = 3(K_1 + K_2)\varepsilon_{\text{vol}} - 3K_1\varepsilon_{\text{vol}}^{\text{v}} \tag{48b}$$

which will be combined with the rate laws for $\boldsymbol{\varepsilon}_{\text{dev}}$ and ε_{vol} according to eqns (45a) and (b). Furthermore, a straightforward elimination of the viscoelastic strains $\boldsymbol{\varepsilon}_{\text{dev}}$ and ε_{vol} from eqns (48a) and (45a), (48b) and (45b), respectively, yields

$$\boldsymbol{\sigma}_{\text{dev}} + (\tau^{\text{G}})^{\alpha^{\text{G}}}\text{D}^{\alpha^{\text{G}}}\sigma_{\text{dev}} = 2G_{(\infty)}\boldsymbol{\varepsilon}_{\text{dev}} + 2G_{(0)}(\tau^{\text{G}})^{\alpha^{\text{G}}}\text{D}^{\alpha^{\text{G}}}\boldsymbol{\varepsilon}_{\text{dev}}, \quad 0 < \alpha^{\text{G}} \leqslant 1 \tag{49a}$$

$$\sigma_{\text{vol}} + (\tau^{\text{K}})^{\alpha^{\text{K}}}\text{D}^{\alpha^{\text{K}}}\sigma_{\text{vol}} = 3K_{(\infty)}\varepsilon_{\text{vol}} + 3K_{(0)}(\tau^{\text{K}})^{\alpha^{\text{K}}}\text{D}^{\alpha^{\text{K}}}\varepsilon_{\text{vol}}, \quad 0 < \alpha^{\text{K}} \leqslant 1 \tag{49b}$$

where $G_{(0)} = G_1 + G_2$ and $G_{(\infty)} = G_2$ are identified as the instantaneous shear modulus and long time shear modulus, respectively, while $K_{(0)} = K_1 + K_2$ and $K_{(\infty)} = K_2$ are identified as the instantaneous bulk modulus and long time bulk modulus, respectively. The simplest possible isotropic model is defined by the same distribution of relaxation times in shear and volumetric responses, which is obtained by setting $\alpha^{\text{G}} = \alpha^{\text{K}} = \alpha$ and $\tau^{\text{G}} = \tau^{\text{K}} = \tau$. In this case only six parameters are required, and eqns (29a) and (b) simplify to

$$\boldsymbol{\sigma}_{\text{dev}} + (\tau)^{\alpha}\text{D}^{\alpha}\boldsymbol{\sigma}_{\text{dev}} = 2G_{(\infty)}\boldsymbol{\varepsilon}_{\text{dev}} + 2G_{(0)}(\tau)^{\alpha}\text{D}^{\alpha}\boldsymbol{\varepsilon}_{\text{dev}} \tag{50a}$$

$$\sigma_{\text{vol}} + (\tau)^{\alpha}\text{D}^{\alpha}\sigma_{\text{vol}} = 3K_{(\infty)}\varepsilon_{\text{vol}} + 3K_{(0)}(\tau)^{\alpha}\text{D}^{\alpha}\varepsilon_{\text{vol}} \tag{50b}$$

Equations (50a) and (b) are often referred to as the fractional calculus model of viscoelasticity, here generalized to 3-D states for isotropic materials, see e.g., Bagley and Torvik (1983). Some work using this fractional calculus model of viscoelasticity to describe viscoelastic damping in the structural equations of motion have been carried out, see, e.g., Bagley and Calico (1991), Fenander (1996). However, when incorporated directly into the structural equations of motion, this form leads to higher order differential equations in time, which demand for initial conditions of fractional order higher than one, see Enelund and Olsson (1995). The present formulation, which involves the concept of internal variables, will lead to well-posed initial value problems and only require initial conditions for the physical quantities, when incorporated into the structural equations of motion. Thus, the physical interpretation and verification of the fractional order initial conditions are avoided.

4. Numerical integration of constitutive response

4.1. Anisotropic model

Since the integer order rate law ($\alpha_{k(\beta)} = 1$) is obtained as a special case of the more general fractional order rate laws, we consider the latter case. A suitable truncation of Grünwald's definition of fractional differentiation yields a possible approximation for the fractional time rate of a function $y(t)$ as (see Oldham and Spanier, 1974)

$$^{n+1}(\mathrm{D}^{\alpha}y) = \frac{1}{(\Delta t)^{\alpha}}\sum_{j=0}^{n} b_j(\alpha)^{n+1-j}y, \quad \text{with} \quad b_j(\alpha) = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} \tag{51}$$

It is then assumed that the spacing in time is uniform, i.e., $^{n}y(t) = y(n\Delta t)$. The calculation of $b_j(\alpha)$ is simplified by the recursion formula

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} = \frac{(j-1-\alpha)}{j}\frac{\Gamma(j-1-\alpha)}{\Gamma(j)} \tag{52}$$

Note that no evaluation of gamma functions are needed whatsoever and the coefficients $b_j(\alpha)$ are given by

$$b_0(\alpha) = 1, \quad b_1(\alpha) = -\alpha, \ldots, b_k(\alpha) = \frac{(k-1-\alpha)}{k}b_{k-1}(\alpha), \ldots$$

For convenience we rewrite the expression in eqn (51) as

$$^{n+1}(\mathrm{D}^{\alpha}y) = \frac{1}{(\Delta t)^{\alpha}}(^{n+1}y - {}^{n}\bar{y}), \quad \text{with}\, ^{n}\bar{y} = -\sum_{j=1}^{n} b_j(\alpha)^{n+1-j}y \tag{53}$$

where $^{n}\bar{y}$ is a known quantity at time t_{n+1} (and thus plays the role of ^{n}y when $\alpha = 1$). In order to obtain (53), it was used that $b_0(\alpha) = 1$.

We are now in the position to apply the General Midpoint rule to eqn (19), while using eqn (53) and combining with eqn (11), to obtain the updated dissipative stress quantities $^{n+1}Q_{k(\beta)}$. The result becomes

$$^{n+1}Q_{k(\beta)} = {}^{n}Q^{\mathrm{v}}_{k(\beta)} + \alpha_{k(\beta)}\Delta Q^{\mathrm{e}}_{k(\beta)}$$

with

$$a_{k(\beta)}(\alpha_{k(\beta)}, \theta, \Delta t) = \left(1+\theta\left(\frac{\Delta t}{\tau_{k(\beta)}}\right)^{\alpha_{k(\beta)}}\right)^{-1} \tag{54}$$

where

$$ {}^{n}Q^{\mathrm{v}}_{k(\beta)} = \alpha_{k(\beta)}\left[1-(1-\theta)\left(\frac{\Delta t}{\tau_{k(\beta)}}\right)^{\alpha_{k(\beta)}}\right]{}^{n}Q_{k(\beta)} + \alpha_{k(\beta)}E_{k(\beta)}({}^{n}q_{k(\beta)} - {}^{n}\bar{q}_{k(\beta)}) \tag{55} $$

and $\Delta Q^{\mathrm{e}}_{k(\beta)}$ is the incremental elastic stress defined as

$$ \Delta Q^{\mathrm{e}}_{k(\beta)} = E_{k(\beta)}\boldsymbol{\varphi}_{\beta}\colon \Delta\boldsymbol{\varepsilon} \tag{56} $$

Clearly, the choice of θ ($\theta \in [0, 1]$) defines the implicitness of the integration rule in standard fashion; e.g., the classical midpoint rule is defined by $\theta = 1/2$, whereas the Backward Euler rule is defined by $\theta = 1$.

Now, upon inserting ${}^{n+1}Q_{k(\beta)}$ from eqn (54) and eqns (6) and (12), we obtain the updated stress ${}^{n+1}\boldsymbol{\sigma}$ as

$$ {}^{n+1}\boldsymbol{\sigma} = \sum_{k=1}^{N}\sum_{\beta=1}^{M} {}^{n+1}Q_{k(\beta)}\boldsymbol{\varphi}_{\beta} = {}^{n}\boldsymbol{\sigma}^{\mathrm{v}} + \mathscr{E}^{\mathrm{v}}\colon \Delta\boldsymbol{\varepsilon} \tag{57} $$

where

$$ {}^{n}\boldsymbol{\sigma}^{\mathrm{v}} = \sum_{k=1}^{N}\sum_{\beta=1}^{M} {}^{n}Q^{\mathrm{v}}_{k(\beta)}\boldsymbol{\varphi}_{\beta} \tag{58a} $$

and

$$ \mathscr{E}^{v} = \sum_{k=1}^{N}\sum_{\beta=1}^{M} a_{k(\beta)}E_{k(\beta)}\boldsymbol{\varphi}_{\beta} \otimes \boldsymbol{\varphi}_{\beta} \tag{58b} $$

We may denote ${}^{n}\boldsymbol{\sigma}^{\mathrm{v}}$ and $\mathscr{E}^{\mathrm{v}}$ the "algorithmic" equivalents to ${}^{n}\boldsymbol{\sigma}$ and $\mathscr{E}^{\mathrm{e}}$, respectively, since they are functions of Δt via $a_{k(\beta)}(\alpha_{k(\beta)}, \theta, \Delta t)$. Clearly, when $\Delta t = 0$ (while $\Delta\boldsymbol{\varepsilon} \neq \mathbf{0}$, corresponding to infinitely rapid loading), then $a_{k(\beta)} = 1$ and it follows from eqns (55), (58a) and (b) that ${}^{n}\boldsymbol{\sigma}^{\mathrm{v}} = {}^{n}\boldsymbol{\sigma}$ and $\mathscr{E}^{\mathrm{v}} = \mathscr{E}^{\mathrm{e}}$.

Remark: In contrast to the use of integer rate laws ($\alpha_{k(\beta)} = 1$), it is necessary to calculate and store all the values ${}^{j}q_{k(\beta)}$ for $j = 0, \beta = 1, 2, \ldots, n$, so that ${}^{n}\bar{q}_{k(\beta)}$ can be calculated from eqn $(53)_2$. Each value of ${}^{-n+1}q_{k(\beta)}$ is most conveniently obtained from the relation in eqn (11) once ${}^{n+1}Q_{k(\beta)}$ has been calculated from eqn (54). In the special case that $\alpha_{k(\beta)} = 1$, we simply set ${}^{n}\bar{q}_{k(\beta)} = {}^{n}q_{k(\beta)}$ and there is no need to store the history.

4.2. Isotropic model

In the special case of isotropy, we obtain in a fashion that is similar to the general situation:

$$ {}^{n+1}\boldsymbol{\sigma}^{\mathrm{v}}_{k,\mathrm{dev}} = {}^{n}\boldsymbol{\sigma}^{\mathrm{vv}}_{k,\mathrm{dev}} + a^{\mathrm{G}}_{k}\Delta\boldsymbol{\sigma}^{\mathrm{ve}}_{k,\mathrm{dev}} \quad \text{with } a^{\mathrm{G}}_{k}(\alpha^{\mathrm{G}}_{k}, \theta, \Delta t) = \left(1+\theta\left(\frac{\Delta t}{\tau^{\mathrm{G}}_{k}}\right)^{\alpha^{\mathrm{G}}_{k}}\right)^{-1} \tag{59} $$

$$^{n+1}\sigma^{\mathrm{v}}_{k,\mathrm{vol}} = {}^{n}\sigma^{\mathrm{vv}}_{k,\mathrm{vol}} + a^{\mathrm{K}}_{k}\Delta\sigma^{\mathrm{ve}}_{k,\mathrm{vol}} \quad \text{with } a^{\mathrm{K}}_{k}(\alpha^{\mathrm{K}}_{k},\theta,\Delta t) = \left(1+\theta\left(\frac{\Delta t}{\tau^{\mathrm{G}}_{k}}\right)^{\alpha^{\mathrm{G}}_{k}}\right)^{-1} \tag{60}$$

where

$$^{n}\boldsymbol{\sigma}^{\mathrm{vv}}_{k,\mathrm{dev}} = a^{\mathrm{G}}_{k}\left[1-(1-\theta)\left(\frac{\Delta t}{\tau^{\mathrm{G}}_{k}}\right)^{\alpha^{\mathrm{G}}_{k}}\right]{}^{n}\boldsymbol{\sigma}^{\mathrm{v}}_{k,\mathrm{dev}} + 2a^{\mathrm{G}}_{k}G_{k}({}^{n}\boldsymbol{\varepsilon}^{\mathrm{v}}_{k,\mathrm{dev}} - {}^{n}\bar{\boldsymbol{\varepsilon}}^{\mathrm{v}}_{k,\mathrm{dev}}) \tag{61}$$

$$^{n}\sigma^{\mathrm{vv}}_{k,\mathrm{vol}} = a^{\mathrm{K}}_{k}\left[1-(1-\theta)\left(\frac{\Delta t}{\tau^{\mathrm{K}}_{k}}\right)^{\alpha^{\mathrm{K}}_{k}}\right]{}^{n}\sigma^{\mathrm{v}}_{k,\mathrm{vol}} + 3a^{\mathrm{K}}_{k}K_{k}({}^{n}\varepsilon^{\mathrm{v}}_{k,\mathrm{vol}} - {}^{n}\bar{\varepsilon}^{\mathrm{v}}_{k,\mathrm{vol}}) \tag{62}$$

whereas $\Delta\boldsymbol{\sigma}^{\mathrm{ve}}_{k,\mathrm{dev}}$ and $\Delta\sigma^{\mathrm{ve}}_{k,\mathrm{vol}}$ are the incremental elastic stresses defined as

$$\Delta\boldsymbol{\sigma}^{\mathrm{ve}}_{k,\mathrm{dev}} = 2G_{k}\mathscr{I}_{\mathrm{dev}}\colon\Delta\boldsymbol{\varepsilon} = 2G_{k}(\Delta\boldsymbol{\varepsilon})_{\mathrm{dev}} \tag{63}$$

$$\Delta\sigma^{\mathrm{ve}}_{k,\mathrm{vol}} = 3K_{k}\boldsymbol{\delta}\colon\Delta\boldsymbol{\varepsilon} = 3K_{k}(\Delta\varepsilon)_{\mathrm{vol}} \tag{64}$$

Upon inserting eqns (59) and (60) with (63) and (64) into eqn (6), we obtain the updated stress $^{n+1}\boldsymbol{\sigma}$ as

$$^{n+1}\boldsymbol{\sigma} = \sum_{k=1}^{N} {}^{n+1}\boldsymbol{\sigma}^{\mathrm{v}}_{k} = {}^{n}\boldsymbol{\sigma}^{\mathrm{v}} + \mathscr{E}^{\mathrm{v}}\colon\Delta\boldsymbol{\varepsilon} \tag{65}$$

where

$$^{n}\boldsymbol{\sigma}^{\mathrm{v}} = {}^{n}\boldsymbol{\sigma}^{\mathrm{v}}_{\mathrm{dev}} + \frac{1}{3}{}^{n}\sigma^{\mathrm{v}}_{\mathrm{vol}}\boldsymbol{\delta} \quad \text{with } {}^{n}\boldsymbol{\sigma}^{\mathrm{v}}_{\mathrm{dev}} = \sum_{k=1}^{N} {}^{n}\boldsymbol{\sigma}^{\mathrm{vv}}_{k,\mathrm{dev}}, \quad {}^{n}\sigma^{\mathrm{v}}_{\mathrm{vol}} = \sum_{k=1}^{N} {}^{n}\sigma^{\mathrm{vv}}_{k,\mathrm{vol}} \tag{66}$$

whereas

$$\mathscr{E}^{\mathrm{v}} = 2G^{\mathrm{v}}\mathscr{I}_{\mathrm{dev}} + K^{\mathrm{v}}\boldsymbol{\delta}\otimes\boldsymbol{\delta} \quad \text{with } G^{\mathrm{v}} = \sum_{k=1}^{N} a^{\mathrm{G}}_{k}G_{k}, \quad K^{\mathrm{v}} = \sum_{k=1}^{N} a^{\mathrm{K}}_{k}K_{k} \tag{67}$$

4.4. Constitutive response

To characterize the present viscoelastic model we study the uniaxial constitutive creep and relaxation responses in the case of two parallel Maxwell chains with $\tau_1 = \tau$, $\tau_2 \to \infty$ and $E_{(0)}/E_{(\infty)} = 2$. The stress relaxation function (i.e., the uniaxial stress response on an applied unit strain at time $t = 0$) is given by eqn (29). Figure 2 shows the stress relaxation function according to eqn (29) for different orders of fractional differentiation α in the rate laws for the viscous strain. We use the asymptotic expansion of the Mittag–Leffler function (see Enelund and Olsson, 1995; Bateman, 1955) when displaying the stress relaxation function for large times. From Fig. 2 we may interpret the parameters of the present viscoelastic model: the relaxation constant τ gives the transition from short time glassy behavior to the long time rubbery behavior, while the order of fractional differentiation α gives the slope of the relaxation curve between the two regions. Figure 3 shows the numerically obtained constitutive creep function or the creep compliance (i.e., the uniaxial strain response up on a unit stress applied at time $t = 0$) for different values of α. A fully

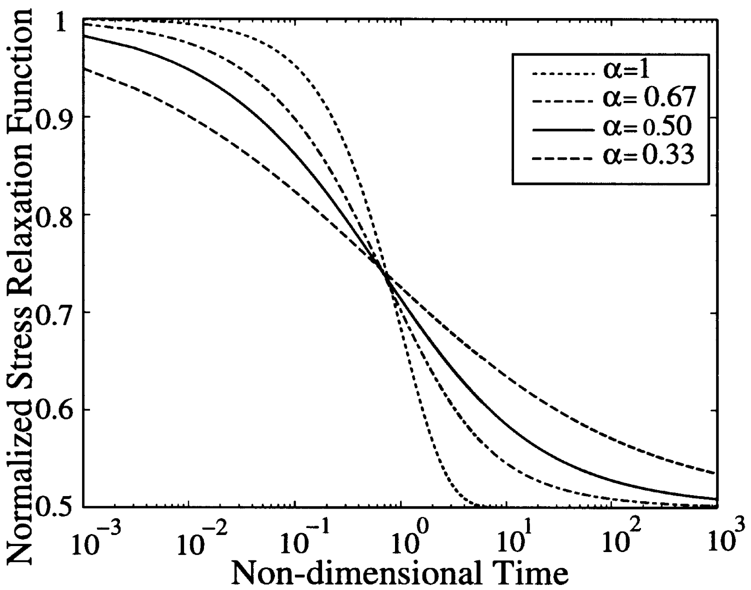

Fig. 2. Normalized stress relaxation function $\sigma_{\text{rel}}/(E_{(0)}$ vs non-dimensional time t/τ. Results are given for $E_{(0)}/E_{(\infty)} = 2$. The influence of different values of the fractional derivative exponent α is shown.

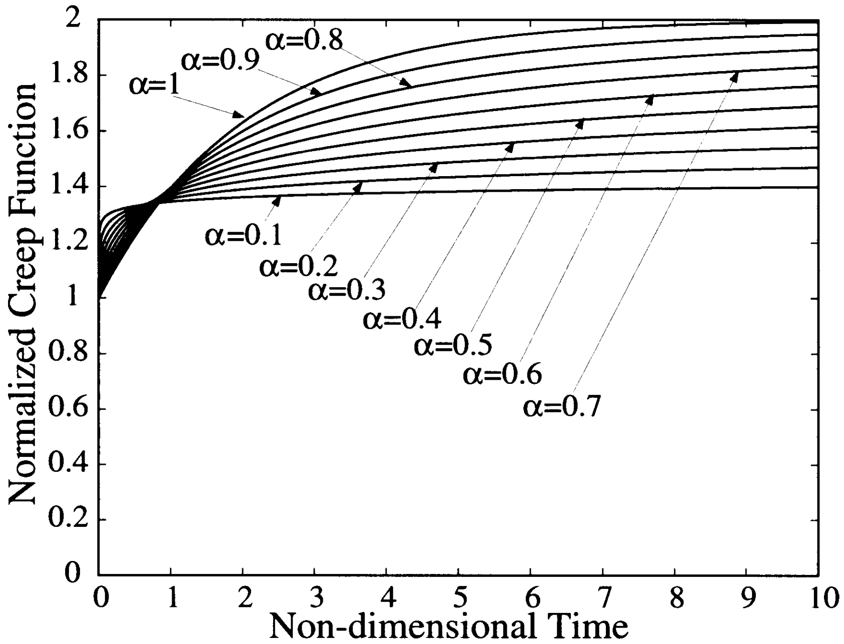

Fig. 3. Normalized creep function $\varepsilon_{\text{creep}}/\varepsilon_0$ vs non-dimensional time t/τ. Here is $\varepsilon_0 = \sigma_0/E_{(0)}$ (the instantaneous strain response). Results are given for $E_{(0)}/E_{(\infty)} = 2$. The influence of the different values of fractional derivative exponent α is shown.

implicit scheme [$\theta = 1$ in eqns (54) and (55)] as outlined in Section 4 is employed. Based on accuracy considerations, we use the time step $\Delta t = 0.01\tau$. The curves in Fig. 3 agree very well with analytical creep functions (not displayed in the figure). The analytical creep function is derived in the same manner as the stress relaxation function, see Enelund (1997). As seen in Figs 2 and 3, stress relaxation and creep over many decades can be modeled by the use of a fractional differentiation operator in the rate laws. Note that the time required for the constitutive responses to reach their long-time asymptotic values increases considerably with decreasing order of the fractional derivative exponent in the rate law. Moreover, the initial or short time ($t/\tau < 1$) relaxation and creep increase most significantly with decreasing order of the fractional derivative exponent α.

5. Structural analysis

With the constitutive theory for general stress states presented in the previous sections, it is possible to analyze the structural responses upon a suitable finite element discretization. Formally, any such set of structural equations takes the form (in matrix notation)

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{f} = \boldsymbol{0} \quad \text{with} \boldsymbol{f} = \boldsymbol{f}^{\text{int}} - \boldsymbol{f}^{\text{ext}} \tag{68}$$

together with initial conditions

$$\boldsymbol{u}(0) = {}^0\boldsymbol{u} \quad \text{and} \quad \dot{\boldsymbol{u}}(0) = {}^0\boldsymbol{v} \tag{69}$$

where $\boldsymbol{u}$ is the nodal displacement vector, $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{f}^{\text{int}}$ is the internal nodal force vector (corresponding to stresses $\boldsymbol{\sigma}$) and $\boldsymbol{f}^{\text{ext}}$ is the external nodal force vector (corresponding to applied load).

Using e.g., a "nearly" explicit Newmark's method ($\beta = 0$ and $\gamma = 1/2$, see Hughes, 1987) with a lumped mass matrix, i.e., $\boldsymbol{M} \to \boldsymbol{M}_{\text{lump}} = \text{diag}\,[m_1, m_2, \ldots, m_L]$, we obtain the nodal displacement solution ${}^{n+1}u_i$ from

$${}^{n+1}u_i = {}^n u_i + \Delta t\, {}^n v_i - \frac{\Delta t^2}{2} m_i^{-1}\, {}^n f_i, \quad i = 1, 2, \ldots, L \tag{70}$$

The updated element spatial distribution of strain ${}^{n+1}\boldsymbol{\varepsilon}^{(m)}$ can now be evaluated as

$${}^{n+1}\boldsymbol{\varepsilon}^{(m)} = \boldsymbol{B}^{(m)}\, {}^{n+1}\boldsymbol{u}^{(m)} \tag{71}$$

where $\boldsymbol{B}^{(m)}$ is the element strain–displacement matrix and $\boldsymbol{u}^{(m)}$ is element nodal displacements. By using the updated strains (and the previous histories), the updated element spatial distribution of the stress ${}^{n+1}\boldsymbol{\sigma}$ within each element can be obtained by using the scheme for numerical integration of the constitutive response described in Section 4 [see e.g., eqn (65)]. With this solution, ${}^{n+1}f_i^{\text{int}}$ can be calculated, whereby the element spatial distribution of stress ${}^{n+1}\boldsymbol{\sigma}^{(m)}$ is employed:

$${}^{n+1}\boldsymbol{f}^{\text{int}} = \sum_{m=1}^{NEL} \int_{V^{(m)}} \boldsymbol{B}^{(m)}\, {}^{n+1}\boldsymbol{\sigma}^{(m)}\, \mathrm{d}V \tag{72}$$

where NEL is the number of elements, V_j is the element volume and $\boldsymbol{B}_j$ is the (element) strain–displacement matrix. When ${}^{n+1}f_i^{\text{int}}$ is known, the nodal velocities ${}^{n+1}v_i$ are obtained from

$$ {}^{n+1}v_i = {}^{n}v_i - \frac{\Delta t}{2} m_i^{-1}({}^{n+1}f_i + {}^{n}f_i) \tag{73} $$

and a new time step can be taken.

6. Numerical examples

The present general three-dimensional viscoelastic material model with integer and fractional order rate laws for the pertinent internal variables and the corresponding algorithm for integration of the constitutive response is implemented, as user-supplied material routines, in the commercial FE-codes ABAQUS-Standard see ABAQUS/Standard Version 5.4 (1994) and ABAQUS-Explicit see ABAQUS/Explicit Version 5.5 (1995). ABAQUS-Standard is used for calculations of quasistatic responses (i.e. where no inertia forces are considered). The corresponding material routine uses a fully implicit scheme ($\theta = 1$) for time integration the updated stress [cf eqn (54)]. ABAQUS-Explicit is used for calculations of dynamic responses. An explicit scheme ($\theta = 0$) together with the explicit difference scheme provided by ABAQUS-Explicit for the time integration of the system of equations on structure level is then used. Algorithmic damping is not imposed in the explicit central difference scheme for the time integration of the structural responses. The appearances of these numerical implementations will be demonstrated in a few examples.

6.1. Dynamic response of viscoelastic bar

The first example considers a horizontal fixed-free viscoelastic uniform bar subjected to a step load applied at time $t = 0$. The amplitude of the load is equal to 1 N. The simplest solid linear viscoelastic material model (i.e., two parallel Maxwell chains and a fractional order rate law for the pertinent internal variable) is used. The following data are used (see also Enelund and Josefson, 1997):

Length $= 0.5$ m, cross-sectional area $= 0.0025$ m^2, $E_{(0)} = E_1 + E_2 = 10$ MN/m^2, $E_{(\infty)} = E_2 = 7$ MN/m^2, density $= 1000$ kg/m^3, $\tau_1 = \tau = 0.02$ s, $\tau_2 = \infty$ and $\alpha = 0.5$.

Figure 4 shows the calculated tip displacements of the bar. To investigate the influence of the numerical modeling, four different FE-models were used: the first one is modeled by five linear two-node bar elements of equal length. The second one is modeled with five linear two-node bar elements of unequal length, the ratio between the longest and shortest element is four to one. The third one is modeled with five equal rectangular four-node plane stress elements. The fourth one is modeled by five equal four-node plane stress elements but distorted so that the "vertical" sides form a 45° angle with the horizontal direction of the bar. The three first models give about the same result, while the long time response for the strongly distorted 2-D mesh is wrong in both amplitude and period length. The result from the bar model with equal elements gives the same result as in Enelund and Josefson (1997). Note that the time-integration of the viscoelastic constitutive law is carried out in a different manner. This solution is (in its turn) verified by comparing

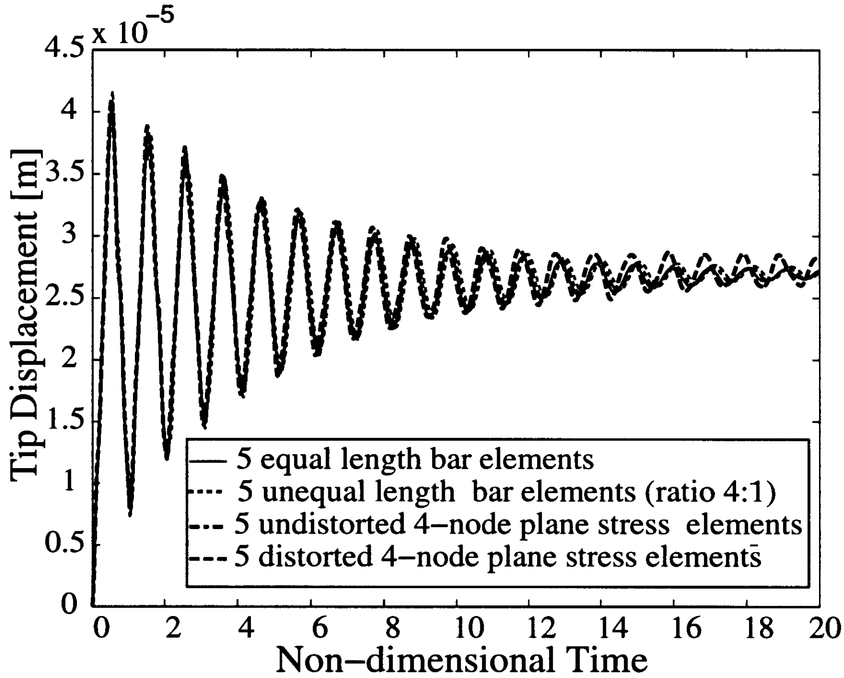

Fig. 4. Tip displacements of a step loaded viscoelastic bar modeled with equal and unequal bar elements and undistorted and distorted plane stress elements vs non-dimensional time t/τ.

with a time series expansion of the analytical solution. The four-node elements are the standard 2-D plane stress elements in ABAQUS-Explicit. For these elements only one integration point in the Gaussian quadrature is employed for element stiffness calculations. Moreover, hourglass control and lumped mass matrix are also used. The explicit central difference scheme is only conditionable stable and the critical time step in the elastic case is $\Delta t_{\mathrm{crit}} = 2/\omega_{\max}$, where $\omega_{\max}$ is the highest eigenfrequency of the FE-model. Numerical experiments in Enelund and Josefson (1997) indicate that the stability limit for the elastic case (using instantaneous material parameters) seems to be a good estimation of the stability limit also for the viscoelastic case. The critical time step used in ABAQUS-Explicit is estimated from a CFL-condition involving typical element length and a wave propagation velocity (see Cook et al., 1989; ABAQUS/Explicit Version 5.5, 1995). ABAQUS-Explicit checks and adjusts the critical time step before taking a new step. In all (explicit) examples, we prescribe a constant time step smaller than the critical time step and we have not noticed any enforced adjustments of the critical time step during the calculations in ABAQUS-Explicit.

6.2. *Quasistatic response of ballast material*

The second example considers a rigid railway sleeper on a viscoelastic ballast material. The ballast material is resting on an elastic soil layer of infinite extent and for comparison also (in a separate case) on a smooth rigid surface. The isotropic viscoelastic material model with two parallel Maxwell chains ($\tau_2 \rightarrow \infty$) and the same distribution of relaxation times in shear and volumetric

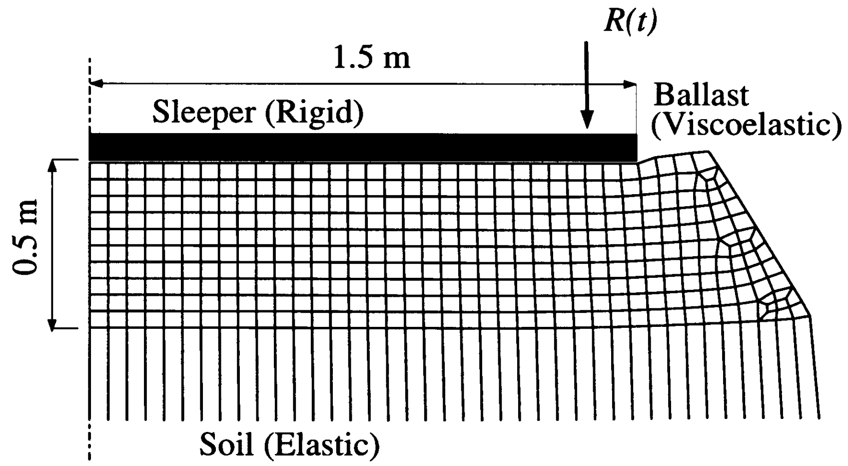

Fig. 5. Geometry and magnified deformed mesh for a rigid sleeper on viscoelastic ballast material resting on a soil layer of infinite extent. Deformed mesh at the end of the time interval is displayed.

responses is used to model the fictitious ballast material. The following material parameters are then used:

$$G_{(0)} = G_1 + G_2 = 86 \text{ MN/m}^2,\ G_{(\infty)} = G_2 = 50 \text{ MN/m}^2,\ K_{(0)} = K_1 + K_2 = 400 \text{ MN/m}^2,$$
$$K_{(\infty)} = K_2 = 233 \text{ MN/m}^2,\ \tau_1^K = \tau_1^G = 0.02\,\text{s},\ \tau_2^K = \tau_2^G = \infty,\ \alpha^K = \alpha^G = \alpha \in (0, 1],$$
$$\text{Density} = 3000 \text{ kg/m}^3.$$

For the elastic soil we use: $E = 40$ MN/m^2, $\nu = 0.4$ and density $= 2000$ kg/m^3. The ballast material is modeled with 382 four-node 2-D elements in plane strain and the soil is modeled with 39 semi-infinite plane strain elements in ABAQUS-Standard. Figure 5 shows the FE-discretization and the magnified deformed mesh of the ballast model at the end of the time interval under consideration. The sleeper is subjected to a quasistatic step load at time $t = 0$. The amplitude of load is 110 kN which corresponds to half the axle load for a typical railway vehicle. Figure 6 shows the quasistatic response at the mid point of the rigid sleeper in the system in Fig. 5. The influence of some different values of the order of fractional differentiation in the rate laws is shown. Figure 7 shows the quasistatic response at the mid point of the sleeper on the viscoelastic ballast material resting on a smooth rigid ground. The curves displayed in Figs 6 and 7 corresponds to the standard relaxation curves often shown for viscoelastic materials (cf Fig. 2).

The time step chosen in both figures is very small ($\Delta t = 0.001\tau$) in order to capture the behavior at very short times. Still, although some 10,000 time increments are used, the short and long time asymptotes for $\alpha = 0.5$ were not closely reached. However, one finds that the short and long time asymptotes for $\alpha = 1$ corresponds to the elastic responses using elastic material parameters $K = K_{(0)}$, $G = G_{(0)}$ and $K = K_{(\infty)}$, $G = G_{(\infty)}$, respectively. Further, with proper choices of time steps, the curves for $\alpha = 0.67$ will approach the short and long time asymptotes.

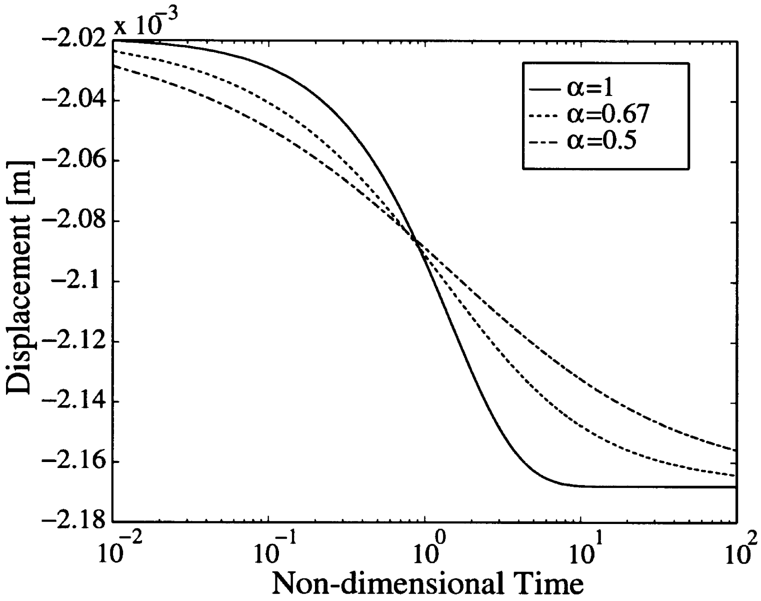

Fig. 6. Quasistatic response at the middle of a rigid sleeper on a viscoelastic ballast material vs non-dimensional time t/τ. The ballast material is resting on an elastic soil layer of infinite extent. The influence of some different values of the fractional derivative exponent α is shown.

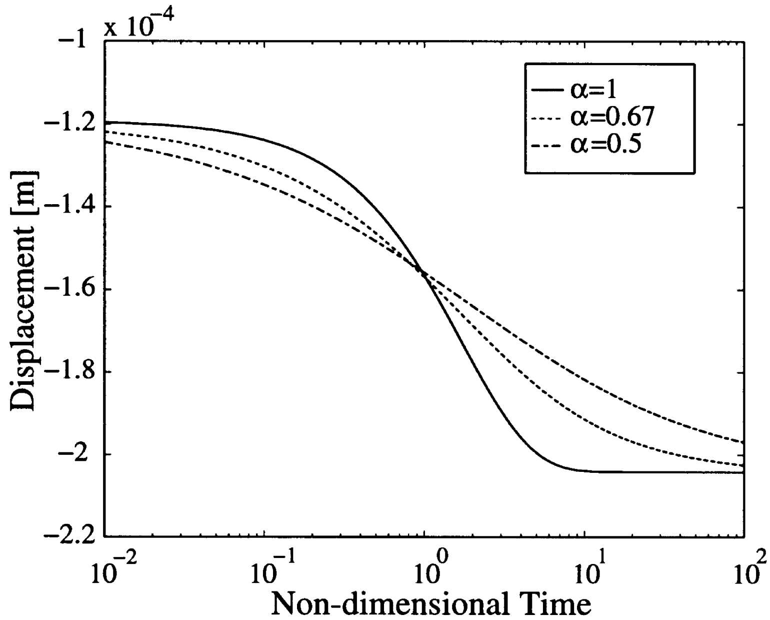

Fig. 7. Quasistatic response at the middle of a rigid sleeper on a viscoelastic ballast material vs non-dimensional time t/τ. The ballast material is resting on smooth rigid ground. The influence of different values of the fractional derivative exponent α is shown.

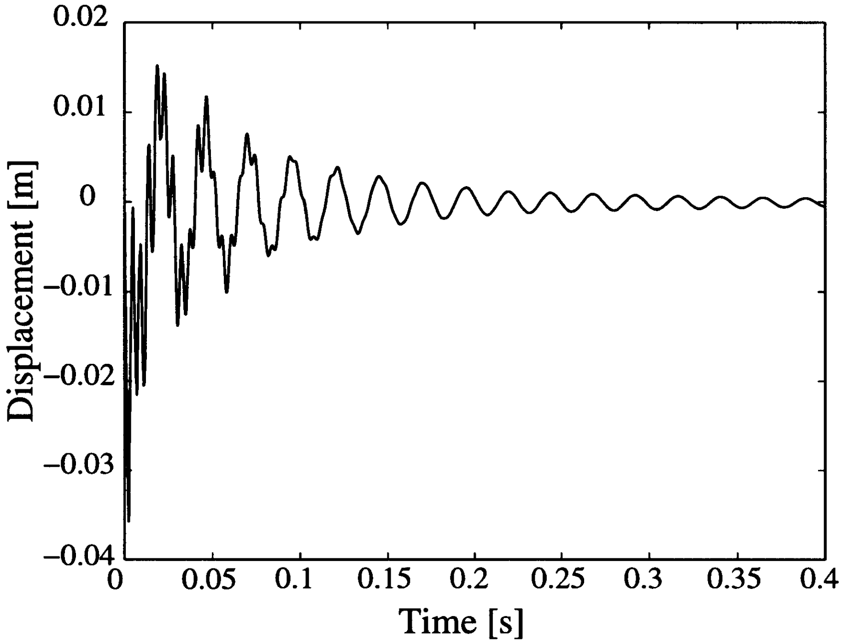

Fig. 8. Dynamic response at the mid point of a rigid sleeper on a viscoelastic ballast material having a fractional derivative exponent equals 0.5 in the rate laws for the viscous strain. The sleeper is subjected to an impulse load that models the overrolling of high-speed train.

6.3. Dynamic response of ballast material

The third example considers the dynamic response of the sleeper-ballast-system resting on smooth rigid ground as described above. The fractional derivative exponent is now taken to be equal to 0.5. First we consider an impulse load modeling the overrolling of a high speed train with a velocity of 300 km/h. The duration of the impulse is 0.0006 s which corresponds to 50 time increments in the integration of the structural response. The magnitude of the impulse is 110 kN. Figure 8 shows the transient response at the mid point of the sleeper in the case of the load described above. In Fig. 9 the transient response at the mid point of the sleeper subjected to a "dynamic" step load applied at time $t = 0$ with magnitude of 110 kN is displayed. The curves in Figs 8 and 9 are obtained by using the full strain tensor history for the viscoelastic ballast material in the material routine for the updated stress [cf eqns (53)–(57)]. Clearly, for large structures and long time intervals it is advantageous if the time series for the strain tensor could be truncated, otherwise the time series of the strains for all Gaussian point of the FE-model should be stored and used in each time increment. Figure 10 displays the same step load response as in Fig. 9 vs time increments when only the strains in the last 1000, 5000 and 10,000 time increments are stored and included when calculating the updated stress. The complete time history corresponds to 15,000 time increments. It is clearly seen that in order to have qualitatively good long time response, the major part of the time history must be retained.

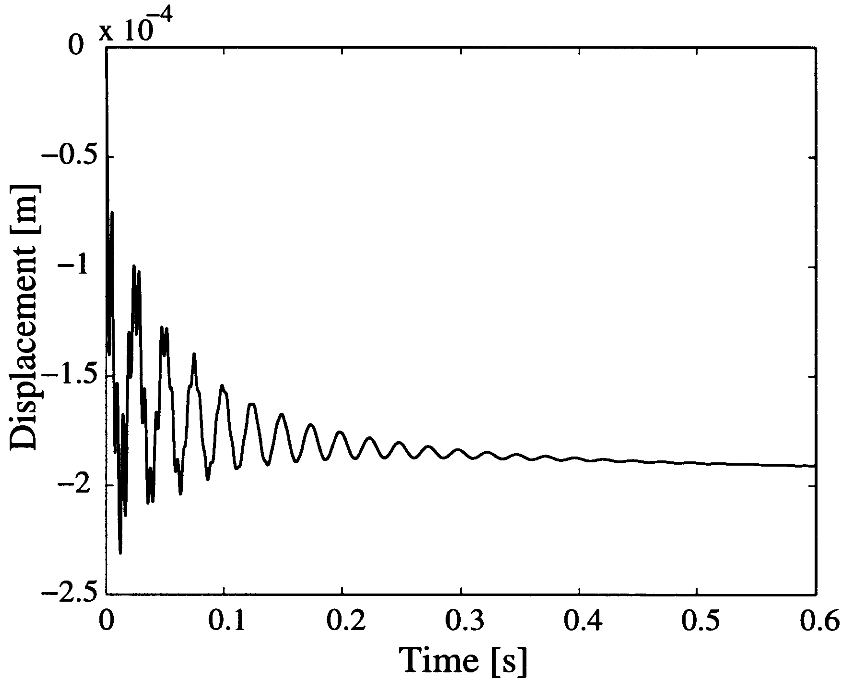

Fig. 9. Dynamic response at the mid point of a rigid sleeper on a viscoelastic ballast material having a fractional derivative exponent equals 0.5 in the rate laws for the viscous strain. The sleeper is subjected to step load with a magnitude equal to 110 kN.

6.4. *Moving load simulating over-rolling of a high speed train*

The fourth example considers a moving load that simulates the effect of a high speed train on a railway ballast material resting on a soil layer. The sleepers (concrete) and the rail (steel) are modeled as being elastic (see Fig. 11). The ballast material is the same viscoelastic material as in previous examples ($\alpha = 0.5$) and the soil layer is regarded as elastic. Used material parameters are given in Section 6.3. The vertical boundaries of the section are absorbing boundaries (non-reflecting) available in ABAQUS-Explicit. The soil layer is resting on a rigid smooth ground. The ballast material is modeled with 240 four-node 2-D elements in plane strain, the soil layer is model with 1680 four-node 2-D elements in plane strain, each sleeper is modeled with one four node 2-D element in plane strain and the rail is modeled with 51 two-node plane Euler–Bernoulli beam elements in ABAQUS-Explicit (see Fig. 11). The system is subjected to a moving impulse load of magnitude 110 kN and with a velocity of $v = 300$ km/h. The duration of the load impulse is 0.0024 s which corresponds to 130 time increments. The load reaches the beginning of the modeled section at time $t = 0$ and reaches the end at time $t = 0.06$ s. Figure 11 shows the geometry and magnified deformed mesh at time $t = 0.024$ s for the model subjected to the moving load as described above. Figure 12 displays the displacement response of the fifth sleeper from the left (see Figure 11) vs time. The time interval studied in Fig. 12 is three times the time required for the load to pass the section.

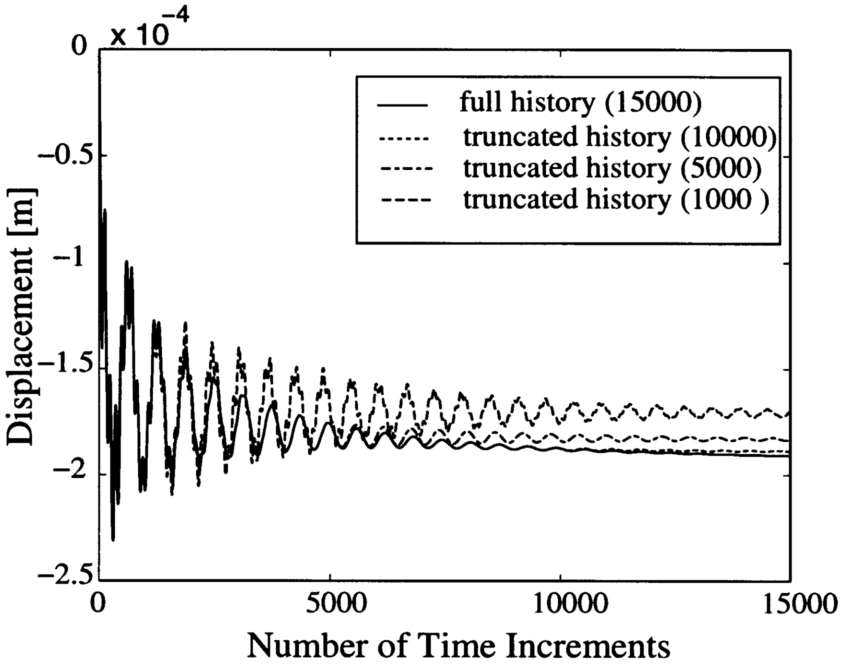

Fig. 10. Dynamic response of the step loaded sleeper. The influence of truncation of the strain time history is shown. The continuous line represents the displacement solution when the full time history of the strain is used. The dotted line represents the displacement solution when a maximum of 10,000 time increments are included in the strain history and the dash-dotted line represents the displacement solution when a maximum of 5000 time increments are included, while the dashed line represents the solution when a maximum of 1000 time increments are included.

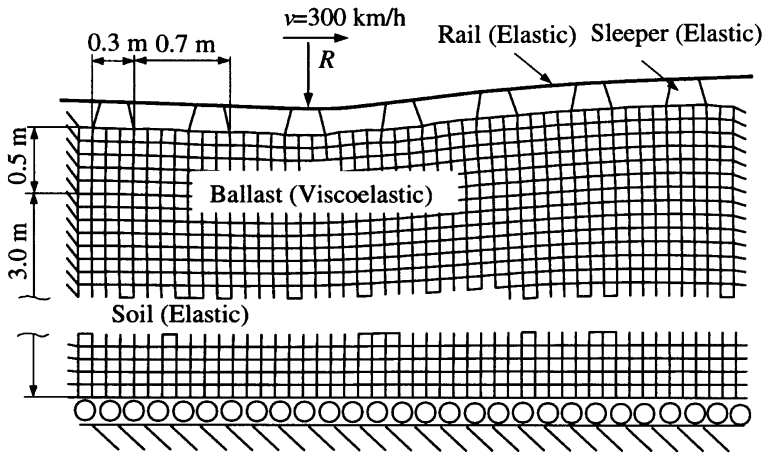

Fig. 11. Geometry and magnified deformed mesh for railway track consisting of elastic sleepers and rails on a viscoelastic ballast material resting on an elastic soil layer. The system is subjected to a moving impulse load modeling a high-speeding train travelling with $v = 300$ km/h. The deformed mesh displayed is valid for time $t = 0.024$ s. It takes the load 0.06 s to pass the section. The vertical boundaries are absorbing boundaries.

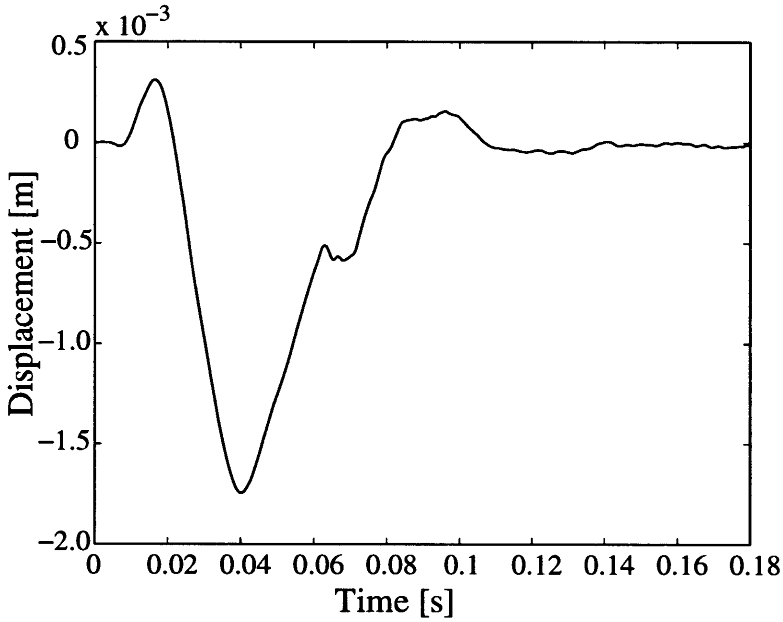

Fig. 12. Dynamic response of a sleeper in a railway track subjected to a moving impulse with velocity $v = 300$ km/h and a magnitude of 110 kN. The impulse reaches the region at time $t = 0$ and it takes 0.06 s for it to pass the section. The ballast material is viscoelastic with a fractional derivative exponent equal to 0.5 in the rate laws for the pertinent internal variables.

7. Summary and discussion

A physically sound three-dimensional anisotropic formulation of the Standard Linear Viscoelastic Solid with integer as well as fractional order rate laws is presented. It is shown that the internal variables can be expressed in terms of the total strain tensor as convolution integrals with singular kernels of Mittag–Leffler function type. A time integration scheme, based on the Generalized Midpoint rule together with the Grünwald algorithm for numerical fractional differentiation, for integration of the constitutive response is developed. This integration scheme can be used together with an explicit difference scheme for calculating structural responses.

By using fractional derivative operators in the rate laws instead of integer order derivative operators, the linear viscoelastic model becomes very flexible for describing structural responses like creep, relaxation and energy dissipation.

The present general viscoelastic model is implemented in a general purposed FE-code together with the algorithm for time integration of constitutive response. One may note that the capability to model viscoelastic material governed by fractional order constitutive relations in general purpose FE-codes is so far very limited. One of the very few examples found in the literature is in Padovan (1987). However, the 2-D-example presented in Padovan (1987) involves a somewhat simpler constitutive model where the derivatives on the stress is omitted [cf eqns (49a) and (b)].

When employing fractional order rate laws, it is found that the complete history of the strain variables has to be sorted and used in each time increment during the FE-analysis. The need to store the strain history depends on how strong the "memory" is and the time needed for material to reach its lower asymptote compared to the time step taken in the dynamic analysis. As seen in Figs 2 and 3, the memory of the viscoelastic material becomes stronger for decreasing value of the fractional derivative exponent α. Lower values of α, like the value $\alpha = 0.5$ used in Fig. 10, increase the need for the use of the complete strain history in the FE-analysis. On the other hand, for higher values of α, closer to unity, a large part of the strain history can be truncated. This means that the viscoelastic model proposed may require to much disc memory when the fractional order rate laws are used for large parts of the structure studied. However, in many engineering applications, only parts of a structure are strongly damped and need to be accurately modeled.

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